

ON THE TODA SYSTEMS OF VHS TYPE

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ABSTRACT. We consider the Toda systems of VHS type with singular sources and provide a criterion for the existence of solutions with prescribed asymptotic behaviour near singularities. We also prove the uniqueness of solution. Our approach uses Simpson's theory of constructing Higgs-Hermitian-Yang-Mills metrics from stability.

1. INTRODUCTION

Let M be a compact Riemann surface and $g = ds^2$ be a smooth riemannian metric on M , and denote the Laplacian associated to g by Δ_g and the Gaussian curvature of g by K_g . For $\epsilon = \pm 1$, we consider systems of partial differential equations of the following form

$$(1.1) \quad \epsilon \begin{pmatrix} \frac{1}{4}\Delta_g u_1 - \frac{K_g}{2} \\ \frac{1}{4}\Delta_g u_2 - \frac{K_g}{2} \\ \vdots \\ \frac{1}{4}\Delta_g u_n - \frac{K_g}{2} \end{pmatrix} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \begin{pmatrix} e^{u_1} \\ e^{u_2} \\ \vdots \\ e^{u_n} \end{pmatrix}$$

on M with finitely many points removed. We call it a Toda system of VHS type or hermitian type according to $\epsilon = 1$ or -1 . Here VHS stands for (polarized complex) variation of Hodge structure. The reason of the name will be clear later. Let $\mu = (\mu_1, \dots, \mu_n)$ be an n -tuple of functions defined on M each of which vanishes at all but finitely many points. The μ_j s are called singular strengths and $S := \{p \in M : \mu_j(p) \neq 0 \text{ for some } j\}$ the set of punctures.

Definition 1.1. A μ -admissible solution to (1.1) is an n -tuple of real-valued smooth functions (u_1, \dots, u_n) on $M \setminus S$ satisfying (1.1) and behaving near the punctures in the following manner: for each $p \in S$ there exists a sufficiently small coordinate chart (U, z) centered at p and smooth bounded functions v_j on U such that

$$u_j = 2\mu_j(p)\log|z| + v_j,$$

on $U \setminus \{p\}$, $j = 1, \dots, n$.

The Toda systems we consider here are usually called type A , manifesting its relation to A_{n+1} . One can consider more general types of Toda systems by replacing the Cartan matrix of type A by those of other types. For the case of smooth solutions on M , i.e. $\mu = (0, \dots, 0)$, there have appeared many studies relating Toda systems with harmonic maps and Higgs bundles, for example, [1] and [3]. The case with nontrivial singular sources is more involved and is the subject of the current paper.

Despite the formal similarity, these two types of Toda systems are quite different in nature, both analytically and geometrically. The VHS type is related to the notion of stability and the hermitian type is more related to the consideration of harmonic maps. In this paper, after developing a general geometric formalism of these systems, which is similar for both types, we will focus on Toda systems of VHS type and treat those of hermitian type in another paper.

Our main result is the following theorem (Theorem 4.2).

Theorem. *For any assignment of singular strengths $\mu = (\mu_1, \dots, \mu_n)$ there exists at most one μ -admissible solution to the Toda system of VHS type. There exists a μ -admissible solution (u_1, \dots, u_n) to the Toda system of VHS type if and only if*

$$d_{n-l+1} + \dots + d_n < l(n-l+1)(\text{genus}(M) - 1),$$

$l = 1, \dots, n$, where

$$d_k := \sum_{p \in M} \left(-\frac{1}{n+1} \sum_{j=1}^n (n-j)\mu_j(p) + \sum_{j=1}^k \mu_j(p) \right),$$

$k = 1, \dots, n$.

The arrangement of this paper is as follows. In Section 2, we introduce the notion of complex pre-VHS and diagonality and show that every Toda system in the above sense corresponds to a special type of complex variation of Hodge structure over the punctured Riemann surface $X = M \setminus S$ whose underlying bundle is a canonically chosen smooth hermitian vector bundle (V, h) . If the Gauss-Manin connection preserves the hermitian form $h(C \cdot, \cdot)$ (C being the Weil operator), then the Toda system is of VHS type (the traditional polarization); if instead it preserves the hermitian metric h , the Toda system is of hermitian type. In Section 3, we recall the notion of Higgs bundle and introduce a Higgs bundles E on X related to a system of VHS type. It is the restriction of a natural vector bundle \tilde{V} on M .

In Section 3, we establish a correspondence between n -tuples of functions on $M \setminus S$ which behave suitably near the punctures and some type of hermitian metrics on E with corresponding asymptotic property. The procedure mimics that of establishing a correspondence between complex variations of Hodge structure and system of Hodge bundles as in [5], simply dropping the

flatness requirement on curvature. Under this correspondence, solutions to Toda systems correspond to flat metrics.

Our main tool in getting the criterion for existence of solutions is the theory of getting Higgs-Hermitian-Yang-Mills metrics (which are flat in our case) from stability, developed by Hitchin [2] and Simpson [5]. As mentioned earlier, the presence of singularities is the main difficulty to be overwhelmed. In order to deal with singularities, we have to use the results of [5] for quasiprojective curves.

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2. COMPLEX VARIATION OF HODGE STRUCTURE

In this section we relate Toda systems with flat connections. The formalism has appeared in studies of Toda systems on a region of \mathbf{R}^2 , for example, in [3]. The vector bundles underlying the flat connections in these classical situations are mainly trivial bundles. We propose a simple globalization as the preparation for further development.

Let X be a complex manifold and denote the sheaf of germs of smooth functions by $\mathcal{A}_X = \mathcal{A}_X^0$.

Definition 2.1. (1) *A complex pre-variation of Hodge structure (complex pre-VHS for short) $(\{V^{r,s}\}, \nabla)$ of weight an integer n on X consists of*

- (i) *smooth complex vector bundles $V^{r,s}$ over X , $r, s \in \mathbf{Z}$, $r + s = n$, with $V^{r,s} = 0$ for all but finitely many (r, s) and*
- (ii) *a connection $\nabla : \mathcal{A}_X^0(V) \rightarrow \mathcal{A}_X^1(V)$ on $V := \bigoplus_{p+q=n} V^{p,q}$ which only has components of total degree $(1, 0)$ and $(0, 1)$. In other words, $\nabla = \bigoplus_{r,s} \nabla^{p,q}$ with*

$$\nabla^{r,s} : \mathcal{A}_X^0(V^{r,s}) \rightarrow \mathcal{A}_X^{1,0}(V^{r-1,s+1}) \oplus \mathcal{A}_X^{1,0}(V^{r,s}) \oplus \mathcal{A}_X^{0,1}(V^{r,s}) \oplus \mathcal{A}_X^{0,1}(V^{r+1,s-1})$$

for each pair (r, s) . If we write $\nabla = \theta + \nabla^{\text{Hodge}} + \theta'$ where

$$\nabla^{\text{Hodge}} = \bigoplus_{r+s=n} \{\nabla^{\text{Hodge}, r,s} : \mathcal{A}_X^0(V^{r,s}) \longrightarrow \mathcal{A}_X^{1,0}(V^{r,s}) \oplus \mathcal{A}_X^{0,1}(V^{r,s})\},$$

$$\theta := \bigoplus_{r+s=n} \{\theta^{r,s} : \mathcal{A}_X^0(V^{r+1,s-1}) \longrightarrow \mathcal{A}_X^{1,0}(V^{r,s})\},$$

and

$$\theta' := \bigoplus_{r+s=n} \{\theta'^{r,s} : \mathcal{A}_X^0(V^{r-1,s+1}) \longrightarrow \mathcal{A}_X^{0,1}(V^{r,s})\},$$

then it is clear that ∇^{Hodge} is also a connection and θ and θ' are \mathcal{A}_X -linear. θ is called the pre-Hodge field of the complex pre-VHS $(\{V^{r,s}\}, \nabla)$.

(2) A complex pre-VHS $(\{V^{r,s}\}, \nabla)$ is a complex VHS if furthermore the curvature of ∇ is 0. If this is the case, θ is called the Hodge field of the complex VHS.

Definition 2.2. Let $(\{V^{r,s}\}, \nabla)$ be a complex pre-VHS of weight n . A Hodge-polarization (resp. hermitian-polarization) consists of $\{h^{r,s}\}$ where $h^{r,s}$ is a smooth hermitian metric on $V^{r,s}$ for each (r, s) such that if $h := \oplus_{r+s=n} h^{r,s}$ then $h(C\cdot, \cdot)$ (resp. $h(\cdot, \cdot)$) is preserved by ∇ , where C is the Weil operator defined by

$$C|_{V^{r,s}} := i^{r-s} \text{id}_{V^{r,s}} = i^n (-1)^s \text{id}_{V^{r,s}}.$$

More precisely, this means that

$$X\langle\sigma, \tau\rangle = \langle\nabla_X \sigma, \tau\rangle + \langle\sigma, \nabla_{\overline{X}} \tau\rangle,$$

for all $X \in T_x X$, $\sigma, \tau \in \mathcal{A}_X^0(V)_x$, and $\langle\cdot, \cdot\rangle = h(C\cdot, \cdot)$ (resp. $h(\cdot, \cdot)$). If this is the case, we say that $(\{V^{r,s}\}, \nabla)$ is Hodge (resp. hermitian)-polarized by $\{h^{r,s}\}$ (or h for short). It is not hard to see that $\{h^{r,s}\}$ is a Hodge-polarization (resp. hermitian-polarization) if and only if ∇^{Hodge} preserves h and $\theta' = \theta^*$ (resp. $-\theta^*$), where

$$\theta^* : \mathcal{A}_X^0(V) \longrightarrow \mathcal{A}_X^{0,1}(V)$$

is the adjoint of θ with respect to h .

Now we specialize to the situation related to Toda systems. Let M be a Riemann surface and g a riemannian metric on M . Let S be any closed subset of M and $X := M \setminus S$. Let K_M be the canonical line bundle of M . We have a hermitian metric on K_M naturally associated to g : if locally one writes $g = \varphi \cdot \overline{\varphi}$ for some nonvanishing $(1, 0)$ -form φ , then φ and $\overline{\varphi}$ is a unitary frame of K_M . Fix a smooth complex line bundle L such that

$$(2.1) \quad L^{\otimes(n+1)} = K_M^{\otimes \frac{n(n+1)}{2}}.$$

We define

$$(2.2) \quad \tilde{V}^{n-k,k} := L \otimes K_M^{\otimes -k}, k = 0, \dots, n.$$

and equip $\tilde{V}^{n-k,k}$ with the hermitian metrics $\tilde{h}^{n-k,k}$ canonically associated to that of K_M . Let $V^{n-k,k} := \tilde{V}^{n-k,k}|_X$ and $h^{n-k,k} := \tilde{h}^{n-k,k}|_V$. Suppose ∇ is a connection on V such that

- (a) $(\{V^{n-k,k}\}, \nabla)$ is a complex pre-VHS which is Hodge (resp. hermitian)-polarized by $\{h^{n-k,k}\}$ and
- (b) all the corresponding $\theta^{n-k,k}$ are non vanishing.

Remark 2.1. For each k , $\theta^{n-k,k}$ can be viewed as a function b_k on X , since

$$K_M \otimes \text{Hom} \left(L \otimes K_M^{\otimes -k+1}, L \otimes K_M^{\otimes -k} \right)$$

is trivial. Therefore, when talking about the pre-Hodge field of a complex pre-VHS of the form in (2.2) (which is the only kind of complex pre-VHS interesting in the following), we will use (b_1, \dots, b_n) and θ interchangeably.

Let φ and e_L be unitary local frames of K_M and L respectively such that $e_L^{\otimes(n+1)}$ corresponds to $\varphi^{\otimes \frac{n(n+1)}{2}}$. Then

$$e_k := e_L \otimes \varphi^{\otimes -k}$$

is a unitary local frame of $\tilde{V}^{n-k,k}$ for each k , with respect to which we can express the connection form of ∇^{Hodge} as a skew-hermitian trace-free diagonal matrix of 1-forms

$$A^{\text{Hodge}} = \begin{pmatrix} A_0 & & & & \\ & \ddots & & & \\ & & A_k & & \\ & & & \ddots & \\ & & & & A_n \end{pmatrix}$$

and θ as a matrix of $(1,0)$ -forms

$$B = \begin{pmatrix} 0 & & & & \\ B_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & B_n & 0 & \end{pmatrix},$$

where

$$(2.3) \quad B_k = b_k \varphi, \quad k = 1, \dots, n.$$

$\theta' = \epsilon \theta^*$, $\epsilon = \pm 1$ according to the type of polarization, and hence the connection form of ∇ is

$$(2.4) \quad \omega = \begin{pmatrix} A_0 & \epsilon \bar{B}_1 & & & \\ B_1 & A_1 & \epsilon \bar{B}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{n-1} & A_{n-1} & \epsilon \bar{B}_n \\ & & & B_n & A_n \end{pmatrix}.$$

Then the curvature $F_{\nabla} = d\omega + \omega \wedge \omega$, whose (k, l) -entry, $k \leq l$, is

$$\begin{cases} dA_k + \epsilon(B_k \wedge \bar{B}_k - B_{k+1} \wedge \bar{B}_{k+1}), & k = l = 0, 1, \dots, n; \\ dB_k + (A_k - A_{k-1}) \wedge B_k, & k = l + 1 = 1, \dots, n; \\ 0, & \text{otherwise,} \end{cases}$$

with the convention that $B_0 = 0 = B_{n+1}$. Then the complex pre-VHS $(\{V^{n-k,k}\}, \nabla)$ is a complex VHS if and only if $F_{\nabla} = 0$, i.e. locally

$$(2.5) \quad dA_k = -\epsilon(B_k \wedge \bar{B}_k - B_{k+1} \wedge \bar{B}_{k+1}),$$

$k = 0, 1, \dots, n$ and

$$(2.6) \quad dB_k = (A_{k-1} - A_k) \wedge B_k,$$

$k = 1, \dots, n$. Note that $d\varphi = -i\rho \wedge \varphi$ for a unique real 1-form ρ and

$$d\rho = -\frac{i}{2}K_g \varphi \wedge \bar{\varphi}.$$

(2.6) then becomes

$$(2.7) \quad \bar{\partial}b_k - ib_k\rho^{0,1} = (A_{k-1}^{0,1} - A_k^{0,1})b_k,$$

$k = 1, \dots, n$. Since A^{Hodge} is skew-hermitian,

$$(2.8) \quad -\partial\bar{b}_k - i\bar{b}_k\rho^{1,0} = (A_{k-1}^{1,0} - A_k^{1,0})\bar{b}_k,$$

$k = 1, \dots, n$. By the nonvanishing assumption on b_k , (2.7) (or (2.8)) is equivalent to

$$(2.9) \quad A_{k-1} - A_k = \frac{\bar{\partial}b_k}{b_k} - \frac{\partial\bar{b}_k}{\bar{b}_k} - i\rho,$$

$k = 1, \dots, n$. On the other hand, by (2.5) and (2.9), we have

$$(2.10) \quad \partial\bar{\partial} \ln |b_k|^2 - id\rho = \epsilon(-|b_{k-1}|^2 + 2|b_k|^2 - |b_{k+1}|^2)\varphi \wedge \bar{\varphi},$$

$k = 1, \dots, n$, which is equivalent to

$$(2.11) \quad \frac{1}{4}\Delta_g \ln |b_k|^2 - \frac{K_g}{2} = \epsilon(-|b_{k-1}|^2 + 2|b_k|^2 - |b_{k+1}|^2),$$

$k = 1, \dots, n$, namely,

$$(2.12) \quad (u_1, \dots, u_n) := (\ln |b_1|^2, \dots, \ln |b_n|^2)$$

is a solution of the Toda system. Note that

$$(2.13) \quad A_0 + A_1 + \dots + A_n = 0.$$

By (2.9), we have

$$(2.14) \quad A_0 = \frac{1}{n+1} \left(\frac{\bar{\partial}(b_1^n b_2^{n-1} \dots b_n)}{b_1^n b_2^{n-1} \dots b_n} - \frac{\partial(\bar{b}_1^n \bar{b}_2^{n-1} \dots \bar{b}_n)}{\bar{b}_1^n \bar{b}_2^{n-1} \dots \bar{b}_n} \right) - i\frac{n}{2}\rho.$$

Conversely, suppose b_1, \dots, b_n are nonvanishing smooth functions on $X = M \setminus S$ satisfying (2.11) (and hence (2.10)). We can define A_k and B_l locally by the formulas in (2.9), (2.14), and (2.3). Then we put them together to form ω as in (2.4). It is easy to see this local description actually gives a global connection on V . By (2.9), such a connection satisfies (2.6). Finally, (2.11) and (2.13) together imply (2.5). In summary, we have obtained the following statement.

Proposition 2.2. *Let (M, g) be a Riemann surface with a smooth riemannian metric, X an open subset of M , L a smooth complex line bundle with $L^{\otimes(n+1)} = K_M^{\otimes \frac{n(n+1)}{2}}$, $V^{n-k,k} := (L \otimes K_M^{\otimes -k})|_X$, and $h^{n-k,k}$ the metric canonically associated to g , $s = 0, \dots, n$. There exists a one-one correspondence between solutions (b_1, \dots, b_n) to (2.11) with $\epsilon = 1$ (resp. -1) on X and*

connections ∇ on $V = \oplus_k V^{n-k,k}$ such that $(\{V^{n-k,k}\}, \nabla)$ is a complex variation of Hodge structure Hodge (resp. hermitian)-polarized by $\{h^{n-k,k}\}$ with all $\theta^{n-k,k}$ nonvanishing.

It is obvious that two solutions to (2.11) give the same solution (2.12) if their corresponding components differ from each other by a phase, that is a smooth function whose values are complex numbers of unit length. We can rephrase the above proposition as follows.

Proposition 2.3. *Under the assumption of the previous proposition, there exists a one-one correspondence between solutions (u_1, \dots, u_n) to (1.1) with $\epsilon = 1$ (resp. -1) on X and connections ∇ on $V = \oplus_k V^{n-k,k}$ such that $(\{V^{n-k,k}\}, \nabla)$ is a complex variation of Hodge structure Hodge (resp. hermitian)-polarized by $\{h^{n-k,k}\}$ with all Higgs components b_k positive.*

We introduce the following auxiliary notion.

Definition 2.3. *A complex pre-VHS $(\{\tilde{V}^{r,s}\}, \nabla)$ over M is called diagonal if the curvature of ∇ is diagonally valued, i.e.*

$$F_\nabla(Y_1, Y_2)(V_x^{r,s}) \subset V_x^{r,s}, \text{ for all } (r, s), Y_1, Y_2 \in T_x M, x \in M.$$

Proposition 2.4. *For a Riemann surface with smooth metric (M, g) , if we have vector bundles $\{V^{n-k,k}\}$ on $X = M \setminus S$ as in (2.2) with the hermitian metric $\{h^{n-k,k}\}$ canonically associated to g and \mathcal{A}_X -linear maps*

$$\theta^{n-k,k} : \mathcal{A}_X^0(V^{n-k+1,k-1}) \longrightarrow \mathcal{A}_X^{1,0}(V^{n-k,k}),$$

there exists a unique connection ∇ on V making $(\{V^{n-k,k}\}, \nabla)$ a diagonal complex pre-VHS which is polarized by $\{h^{n-k,k}\}$ and whose pre-Hodge field is given by $\{\theta^{n-k,k}\}$.

Proof. By checking the discussion above carefully, the local condition for the diagonality in this situation is precisely (2.6), which implies (2.9), and hence (2.14) by (2.13). Therefore every A_k is determined by all b_s (i.e. by θ). It is also easy to see that the local descriptions patch together well to yield a global connection, which is clearly diagonal. \square

Remark 2.5. *In view of this proposition, we can view the Toda system, or more precisely (2.11), as the condition on a “potential pre-Hodge field” $\theta = (b_1, \dots, b_n)$ such that its uniquely associated diagonal complex pre-VHS is a complex VHS.*

We could have had the entire discussion above on a Riemann surface with metric (X, g) without mentioning any ambient manifold M and the complement S . Later we will come back to the situation in Section 1 that (M, g) be a compact Riemann surface with a smooth metric and

$$S := \{p \in M : \mu_j(p) \neq 0 \text{ for some } j\}$$

is the set of punctures of an assignment of singular strengths μ_j , and this is the reason we chose to state the results in the above manner.

3. THE ASSOCIATED HIGGS BUNDLES OF AN ASSIGNMENT OF SINGULAR STRENGTHS

We recall the definition in [5]. Let X be a complex manifold.

Definition 3.1. A Higgs bundle (E, Φ) consists of

- (i) a holomorphic vector bundle E over M and
- (ii) a holomorphic $\text{End}(E)$ -valued $(1, 0)$ -form Φ , called the Higgs field.

Suppose H is a hermitian metric on E . We will denote the Chern connection of E associated to H by ∇^H and the adjoint of Φ with respect to H by Φ^* . Then $\nabla := \Phi + \nabla^H + \Phi^*$ is a connection on E as well. Typical examples are complex pre-variations of Hodge structure with a Hodge-polarization with $F_{\nabla^{\text{Hodge}}}^{0,2} = 0$. For any such complex pre-VHS there is a canonical structure of holomorphic vector bundle on each $(\{V^{r,s}\})$; it also implies that $(\nabla^{\text{Hodge}})^{0,1} \wedge \theta = 0$, i.e. θ is holomorphic.

In the rest of this section we let M be a compact Riemann surface, $\mu_j, j = 1, \dots, n$ an assignment of singular strengths as in Section 1, and

$$S := \{p \in M : \mu_j(p) \neq 0 \text{ for some } j\}.$$

Definition 3.2. An n -tuple of functions (b_1, \dots, b_n) on $X := M \setminus S$ is of type $\mu = (\mu_1, \dots, \mu_n)$ if all b_k are smooth positive functions and for each $p \in S$, there exists a coordinate chart (U, z) centered at p and positive smooth bounded functions \hat{b}_k on U such that

$$b_k = \hat{b}_k |z|^{\mu_k(p)}$$

on U , $k = 1, \dots, n$.

Now we are going to make some choices which will be fixed through the rest of this section:

- (1) In Section 2 we made use of a complex line bundle L satisfying (2.1).

We will choose a specific one as follows: fix a square-root $K_M^{\frac{1}{2}}$ of K_M

(which exists in any case) and take $L := (K_M^{\frac{1}{2}})^{\otimes n}$.

- (2) Fix a holomorphic atlas of charts $\{(U_\alpha, z_\alpha)\}$ on each of whose member $K_M^{\frac{1}{2}}$ is trivialized by a holomorphic frame σ_α with $\sigma_\alpha^{\otimes 2} = dz_\alpha$. If $\sigma_\beta = \psi_{\alpha\beta} \sigma_\alpha$ on $U_\alpha \cap U_\beta$, then

$$(3.1) \quad \psi_{\alpha\beta}^2 = J_{\alpha\beta} := \frac{dz_\beta}{dz_\alpha}.$$

- (3) For a smooth riemannian metric g on M , when writing $g = \varphi_\alpha \cdot \overline{\varphi}_\alpha$ with a nonvanishing 1-form φ_α in a coordinate chart (U_α, z_α) , we always choose φ_α to be the unique positive multiple of dz_α , i.e. $\varphi_\alpha = \lambda_\alpha dz_\alpha$, $\lambda_\alpha > 0$. We have

$$(3.2) \quad \frac{\lambda_\alpha}{\lambda_\beta} = |J_{\alpha\beta}|$$

on $U_\alpha \cap U_\beta$. In addition, if $d\varphi_\alpha = -i\rho_\alpha \wedge \varphi_\alpha$ for a real 1-form ρ_α , then

$$(3.3) \quad \frac{\bar{\partial}\lambda_\alpha}{\lambda_\alpha} = -i\rho_\alpha^{0,1}.$$

Definition 3.3. For any assignment of singular strengths $\mu = (\mu_1, \dots, \mu_n)$, let $\tilde{V}^{n-k,k}$ be vector bundles (2.2) with the holomorphic structure induced by that of K_M and L chosen above and $\tilde{V} = \oplus_k \tilde{V}^{n-k,k}$.

- (1) Let $E = \tilde{V}|_X$ and $E_k = \tilde{V}^{n-k,k}|_X$. There is a natural Higgs field Φ on V whose components Φ_k correspond to

$$1 \in \Gamma(M, K_M \otimes \text{Hom}(L \otimes K_M^{\otimes -k+1}, L \otimes K_M^{\otimes -k})).$$

- (2) A smooth metric $H = \oplus_k H_k$ on the Higgs bundle (V, Φ) is of type μ if in a sufficiently small coordinate chart (U, z) centered at any puncture p , H_k and $|z|^{\mu_k(p)}$ are mutually bounded with respect to local trivializations of $\tilde{V}^{n-k,k}$.

The key correspondence is given by the following proposition.

Proposition 3.1. There is a one-one correspondence between n -tuple of functions (b_1, \dots, b_n) on $X = M \setminus S$ of type μ and hermitian metrics H of type μ on the Higgs bundle (V, Φ) with $\det H = 1$. Under this correspondence, a solution (b_1, \dots, b_n) to (2.11) corresponds to a metric H with $\nabla = \Phi + \nabla^H + \Phi^*$ flat.

Proof. We start with (b_1, \dots, b_n) of type μ first. By Proposition 2.4, we have the unique diagonal complex pre-VHS $(V := \oplus_k V^{n-k,k}, \nabla)$ (where $V^{n-k,k}$ is the restriction of $\tilde{V}^{n-k,k}$) with pre-Hodge field (b_1, \dots, b_n) polarized by the metric h naturally associated to g . As mentioned at the beginning of this section, this complex pre-VHS determines a Higgs bundle, which will seen to be isomorphic to the Higgs bundle E in Definition 3.3, which has \tilde{V} as a natural holomorphic extension.

In each coordinate chart (U_α, z_α) we select the unitary frame $(e_k)_\alpha := (\sigma_\alpha / |\sigma_\alpha|)^{\otimes n} \otimes \varphi_\alpha^{\otimes -k}$ for $\tilde{V}^{n-k,k}$, $k = 1, \dots, n$. We need to get a holomorphic frame $(e'_k)_\alpha = (\delta_k)_\alpha (e_k)_\alpha$ on $U_\alpha \setminus S$ for each α . This means $(\nabla^{\text{Hodge}})^{0,1}(e'_k)_\alpha = 0$, or equivalently (by (2.13), (2.14), and (3.3))

$$\frac{\bar{\partial}(\delta_k)_\alpha}{(\delta_k)_\alpha} + \frac{\bar{\partial}[(b_1^n b_2^{n-1} \dots b_n)^{\frac{1}{n+1}} (b_1 \dots b_k)^{-1}]}{(b_1^n b_2^{n-1} \dots b_n)^{\frac{1}{n+1}} (b_1 \dots b_k)^{-1}} + \frac{\bar{\partial}\lambda_\alpha^{\frac{n}{2}-k}}{\lambda_\alpha^{\frac{n}{2}-k}} = 0.$$

Therefore, we may take $(\delta_k)_\alpha$ to be $(b_1^n b_2^{n-1} \dots b_n)^{\frac{-1}{n+1}} b_1 \dots b_k \lambda_\alpha^{k-\frac{n}{2}}$ multiplied by any holomorphic function on $U_\alpha \setminus S$. The simplest choice is

$$(3.4) \quad (\delta_k)_\alpha := (b_1^n b_2^{n-1} \dots b_n)^{\frac{-1}{n+1}} b_1 \dots b_k \lambda_\alpha^{k-\frac{n}{2}}.$$

Then $(e'_k)_\beta = \psi_{\alpha\beta}^n J_{\alpha\beta}^{-s} (e'_k)_\alpha$:

$$\begin{aligned} \frac{(e'_k)_\beta}{(e'_k)_\alpha} &= \left(\frac{\lambda_\beta}{\lambda_\alpha} \right)^{k-\frac{n}{2}} \left(\frac{\sigma_\beta}{\sigma_\alpha} \right)^n \left| \frac{\sigma_\beta}{\sigma_\alpha} \right|^{-n} \left(\frac{\varphi_\beta}{\varphi_\alpha} \right)^{-k} \\ &= \psi_{\alpha\beta}^n J_{\alpha\beta}^{-k}. \end{aligned}$$

This shows that E_k is the restriction of $\tilde{V}^{n-k,k}$. We define a smooth metric H_k on E_k by setting on $U_\alpha \setminus S$

$$(3.5) \quad (H_k)_\alpha := |(\delta_k)_\alpha|^2.$$

It is direct to show that this defines a metric on E_k , which is of type μ since

$$(b_1^n b_2^{n-1} \cdots b_n)^{\frac{-1}{n+1}} b_1 \cdots b_k |(f_k)_\alpha|^{-1}$$

has the same order as $|z_\alpha^{d_k(p)}|$ near the punctures $p \in S$ by the assumption that (b_1, \dots, b_n) is of type μ .

Conversely, from a metric of bounded type $H = \oplus_k H_k$ we can get

$$(b_1^n b_2^{n-1} \cdots b_n)^{\frac{-1}{n+1}} b_1 \cdots b_k$$

back from (3.5) and (3.4) and hence can obtain all b_k . This establishes the expected one-one correspondence.

As for the last statement, note that the underlying bundles on both sides of the correspondence are smoothly equivalent over the compliment X the punctures S and the connections on both sides are equivalent under this identification. By Proposition 2.2 the proof is completed. \square

By Proposition 3.1, finding solutions to Toda systems of VHS type with singular strength assignment μ becomes finding hermitian metrics H of type μ on V with $\nabla = \Phi + \nabla^H + \Phi^*$. In the next section we will obtain a criterion of existence of such kind of metrics.

4. STABILITY AND HIGGS-HERMITIAN-YANG-MILLS METRICS

In this section we provide the criterion for the existence of solutions to Toda systems of VHS type and prove their uniqueness. Our main ingredient is Simpson's theory of constructing Higgs-Hermitian-Yang-Mills metrics on Higgs bundles from stability [5]. Let (X, g) be a Kähler manifold, (E, Φ) a Higgs bundle over X and H a smooth hermitian metric on E as in Section 3. Recall that we have set

$$\nabla := \Phi + \nabla^H + \Phi^*.$$

In the following, we will denote the curvature F_∇ of such ∇ induced by H as F_H^{Higgs} . We suppose that (X, g) satisfies suitable assumptions (cf. [5], Section 2, Assumptions 1,2 and 3), which will be fulfilled in our situation (cf. [5], Propositions 2.2 or 2.4).

Definition 4.1. *The metric H is a Hermitian-Yang-Mills metric if the trace-free part of $\Lambda_g F_H^{\text{Higgs}}$ vanishes.*

We will need the notion of stability. Let A be a group acting by bi-holomorphic maps of X preserving the metric g and acting compatibly by automorphisms $a : E \rightarrow E$ preserving the metric K and acting on Φ by homotheties $a\Phi a^{-1} = \lambda(a)\Phi$.

Definition 4.2. ([5], p.877, 878)

- (1) *A sub-Higgs sheaf of a Higgs bundle (E, Φ) is an analytic subsheaf $\mathcal{V} \subset \mathcal{O}(E)$ such that $\Phi : \mathcal{V} \rightarrow \mathcal{O}(K_X) \otimes \mathcal{V}$. (If \mathcal{V} is saturated, outside a set of codimension 2 it is the coherent sheaf associated to a subbundle of E .)*
- (2) *For a saturated subsheaf \mathcal{V} and a smooth metric K on E such that $\sup_X |\Lambda_g F_K^{\text{Higgs}}|_K < \infty$,*

$$\deg(\mathcal{V}, K) := i \int_X \text{Tr } \Lambda_g F_K^{\text{Higgs}}.$$

(This is either a real number or $-\infty$ by [5], Lemma 3.2.)

- (3) *(E, Φ, K) is stable with respect to the A -action if for every proper saturated sub-Higgs sheaf \mathcal{V} preserved by A ,*

$$\frac{\deg(\mathcal{V}, K)}{\text{rk}(\mathcal{V})} < \frac{\deg(E, K)}{\text{rk}(E)}.$$

Our main tool is the following result due to Simpson (cf. [5], Theorem 1 and Proposition 3.3).

Theorem 4.1. (Simpson) *Let (X, g) , (E, Φ) , K , and A satisfy all conditions above. If (E, Φ, K) is stable with respect to the A -action, then there exists a smooth A -invariant Higgs-Hermitian-Yang-Mills metric H with H and K mutually bounded,*

$$\det H = \det K, \text{ and } \bar{\partial}h + [\Phi, h] \in L_{g,K}^2,$$

where h is the unique endomorphism of E such that

$$(\cdot, \cdot)_H = (h(\cdot), \cdot)_K.$$

If furthermore $\Phi \wedge \Phi = 0$, the first Chern form $c_1(E, K) = 0$, and $\int_X c_2(E, K) \wedge \omega_g^{n-2} = 0$, then the connection $\nabla = \Phi + \nabla^H + \Phi^$ is flat. Conversely, if there exists an A -invariant Higgs-Hermitian-Yang-Mills metric, then*

$$\frac{\deg(\mathcal{V}, K)}{\text{rk}(\mathcal{V})} \leq \frac{\deg(E, K)}{\text{rk}(E)}$$

for every proper saturated sub-Higgs sheaf \mathcal{V} preserved by A and equality holds only if $E = \mathcal{V} \oplus \mathcal{V}^\perp$ is an orthogonal direct sum of Higgs subbundles.

Now we give the proof of our main result.

Theorem 4.2. *Let M be a compact Riemann surface with a smooth Riemannian metric g . Let $\mu = (\mu_1, \dots, \mu_n)$ be an assignment of singular strengths and*

$$S := \{p \in M : \mu_j(p) \neq 0 \text{ for some } j\}.$$

There exists at most one μ -admissible solution (u_1, \dots, u_n) to the Toda system of VHS type

$$\begin{pmatrix} \frac{1}{4}\Delta_g u_1 - \frac{K_g}{2} \\ \frac{1}{4}\Delta_g u_2 - \frac{K_g}{2} \\ \vdots \\ \frac{1}{4}\Delta_g u_n - \frac{K_g}{2} \end{pmatrix} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \begin{pmatrix} e^{u_1} \\ e^{u_2} \\ \vdots \\ e^{u_n} \end{pmatrix}.$$

There exists a μ -admissible solution if and only if

$$(4.1) \quad d_{n-l+1} + \dots + d_n < l(n-l+1)(\text{genus}(M) - 1),$$

$l = 1, \dots, n$, where

$$d_k := \sum_{p \in M} \left(-\frac{1}{n+1} \sum_{j=1}^n (n-j)\mu_j(p) + \sum_{j=1}^k \mu_j(p) \right),$$

$k = 1, \dots, n$.

Proof. We prove the statement about uniqueness first. The argument is essentially the same as that in the proof of Lemma 10.9 of [5]. Let (E, Φ) be as in Definition 3.3 (1). Let A be the group $U(1)^{\times(n+1)} \cap SU(n+1)$ acting on X trivially and on E in the obvious diagonal manner. Suppose we have two solutions corresponding to Higgs-Hermitian-Yang-Mills metrics H and H' respectively. Let h be the unique endomorphism of E such that $(\cdot, \cdot)_{H'} = (h(\cdot), \cdot)_H$. Note that h is a positive definite self-adjoint and bounded with respect to H . Taking trace of Lemma 3.1 (c) in [5] gives

$$\Delta_d \text{Tr } h = 2\Delta_{\partial} \text{Tr } h = - \left| (\bar{\partial}h + [\Phi, h])h^{\frac{1}{2}} \right|_H^2 \leq 0.$$

As mentioned above, Assumption 3 in [5] holds for $(X, g|_X)$, and hence a positive bounded subharmonic function must be harmonic. Therefore,

$$\bar{\partial}h + [\Phi, h] = 0,$$

which is equivalent to saying that h is a holomorphic endomorphism of E commuting with Φ . Since h commutes with the A -action, it acts on E diagonally by multiplication with positive numbers h_0, \dots, h_n . The commutativity of h with Φ implies $h_0 = \dots = h_n$. Finally, $h_0 \dots h_n = 1$ since $\det H' = \det H = 1$. This shows that $h = \text{id}_E$ and the proof is completed.

Now let (E, Φ) and E_k be as in Definition 3.3. Suppose that A acts on E diagonally. We equip E with a metric $K := \oplus_j K_j$ of type μ (Definition 3.3) such that $\det K = 1$. Such a metric can be constructed as follows. Choose a covering of M by coordinate disks (U_α, z_α) and a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. We may assume that

- (1) each puncture $s \in S$ is contained in U_α for exactly one α , denoted as $\alpha(s)$, and $z_{\alpha(s)}(s) = 0$;
- (2) for every $s \in S$, there exists an open neighborhood $W_{\alpha(s)} \subset U_{\alpha(s)}$ of s such that $\rho_{\alpha(s)}|_{W_{\alpha(s)}} \equiv 1$;
- (3) every $\tilde{V}^{n-j,j}$ is trivialized on U_α by a holomorphic local frame $e_{j,\alpha}$.

Let σ be an element in the fibre of E_j above $p \in M \setminus S$. We write

$$\sigma = \sigma_\alpha e_{j,\alpha}(p)$$

if $p \in U_\alpha$. Let

$$f_\alpha := \begin{cases} |z_\alpha|^{2d_k(s)}, & \text{if } s \in U_\alpha \cap S; \\ 1, & \text{otherwise.} \end{cases}$$

We define the metric K_j on E_j for $j = 1, \dots, n$ by setting

$$|\sigma|_{K_j}^2 := \sum_{\alpha: p \in U_\alpha} \rho_\alpha(p) f_\alpha(p) |\sigma_\alpha|^2.$$

Finally we let K_0 on E_0 be defined so that $\det K = 1$.

$c_1(E, K) = 0$ by the construction of K . $\Phi \wedge \Phi = 0$ and $c_2(E, K) = 0$ automatically on a Riemann surface.

Since $\dim M = 1$, proper saturated sub-Higgs sheaves are exactly proper holomorphic subbundles of E preserved by Φ . By the form of Φ (a nilpotent string), it is clear that such kind of subbundles are exactly

$$F^l := E_{n-l+1} \oplus \dots \oplus E_n,$$

$l = 1, \dots, n$. By Definition 4.2,

$$\deg(F^l, K) = \sum_{j=n-l+1}^n \deg(E_j, K_j).$$

Note that for each j the smooth metric K_j on E_j can be viewed as a singular metric on $\tilde{V}^{n-j,j}$, whose curvature current represents the first Chern class of $\tilde{V}^{n-j,j} = L \otimes K_M^{\otimes -j}$. By the Poincaré–Lelong formula,

$$\deg \tilde{V}^{n-j,j} = \deg(E_j, K_j) - d_j.$$

Therefore

$$\deg(E_j, K_j) = (\text{genus}(M) - 1)(n - 2j) + d_j,$$

and hence

$$\deg(F^l, K) = (\text{genus}(M) - 1)l(l - n - 1) + d_{n-l+1} + \dots + d_n.$$

It is clear that $\deg(E, K) = 0$. If (4.1) holds for $l = 1, \dots, n$, then (E, Φ, K) is stable. In view of Proposition 3.1 and the first part of Theorem 4.1, we obtain a solution to the Toda system with required asymptotic behaviour near punctures. Conversely, if there exists such a solution, by Proposition 3.1 and the second part of Theorem 4.1, we have

$$d_{n-l+1} + \dots + d_n \leq l(n - l + 1)(\text{genus}(M) - 1),$$

$l = 1, \dots, n$. Actually, all inequalities above are strict since the orthogonal complement of F^l is not a sub-Higgs sheaf, $l = 1, \dots, n$. Therefore (4.1) holds for $l = 1, \dots, n$. \square

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